

Notes on the polynomial identities in random overlap structures

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Abstract

In these notes we review first in some detail the concept of *random overlap structure* (ROSt) applied to fully connected and diluted spin glasses. We then sketch how to write down the general term of the expansion of the energy part from the Boltzmann ROSt (for the Sherrington-Kirkpatrick model) and the corresponding term from the RaMOSt, which is the diluted extension suitable for the Viana-Bray model.

From the ROSt energy term, a set of polynomial identities (often known as Aizenman-Contucci or AC relations) is shown to hold rigorously at every order because of a recursive structure of these polynomials that we prove. We show also, however, that this set is smaller than the full set of AC identities that is already known. Furthermore, when investigating the RaMOSt energy for the diluted counterpart, at higher orders, combinations of such AC identities appear, ultimately suggesting a crucial role for the entropy in generating these constraints in spin glasses.

1 Introduction

The study of mean field spin glasses is very challenging from both a physical [29] and a mathematical [39] point of view. Concerning the latter, an increasing amount of work has, in recent years, developed sophisticated mathematical techniques and used these to confirm several scenarios from theoretical physics (e.g. [9][10][11][19][20][26][31]).

Despite all the results that have been obtained by several techniques that avoid the replica trick (e.g. cavity field [23, 28], stochastic stability [15, 14], stochastic calculus [13, 36] and others [2, 30]) — including, of fundamental importance, the correctness of the Parisi expression for the free energy [25][40] — the question of its uniqueness is still a subject for debate. This brings with it also the question of whether ultrametricity, with all its peculiarities, necessarily holds [29]. Recently, fundamental progress has been made connecting ultrametricity to polynomial identities [5, 32, 33], mainly Ghirlanda Guerra relations (GG) [22], highlighting the importance of polynomial identities in the analysis of mean field spin glasses.

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One of the key approaches in the field is the powerful and physically profound concept of Random Overlap Structures (henceforth ROST) introduced by Aizenman, Sims and Starr in [3].

In this work we want to deepen our understanding of a certain kind of polynomial identities, known as *Aizenman-Contucci* identities (AC) [1]. These characterize, in a sense, the peculiar structure of the spin glass phase (as there is a deep link between AC polynomials and GG identities) [22][34] within the framework of ROST, both for fully connected (Sherrington-Kirkpatrick, SK [23][38]) and for diluted (Viana-Bray, VB [27][41]) systems.

We show how to systematically derive AC relations from the energy contribution of the Boltzman Random Overlap Structure, once a Hamiltonian is given. Interestingly, we find that only a subset of the whole set of known identities can be obtained. Furthermore, when looking at the diluted counterpart, where a Parisi theory has not yet been fully achieved, we show that at high orders of expansion, the AC-like relations come out but combined into larger identities: it is not trivially possible to split them again to show that they are zero separately. In section 2 we introduce the general concept of Random Overlap Structures. Then, in section 3 we show in general terms our technique for finding the desired polynomial identities. In section 4 we apply the idea to the Boltzmann ROST for the SK model, while in section 5 we test it on the Boltzmann RaMOST for the VB case. Section 6 is left for discussion and closes the paper.

2 Random overlap structures

In a nutshell, the ROST generalizes the single spin cavity approach [6][24][23] into one of *several (and possibly many) added* spins. These are in contact with a larger “bath” with its own interaction matrix. The ROST allows the properties of this bath, including the overlaps between different states, to be specified in a very flexible manner by a trial random structure which interacts with the original set of cavity spins. This then permits one to represent the pressure of the SK model as the infimum over a family of such trial structures in a set of probability spaces.

The Parisi ROST [4][37], which has states lying on an ultrametric tree [16][17], has the property of optimality with respect to this principle (i.e. it is one way of realizing the infimum). It is thought to coincide with the (conceptually much simpler) Boltzmann ROST [24] introduced by Guerra, which was shown to share with the former the same optimality.

2.1 Introducing the ROST for the SK

Let us start from a system of $M + N$ spins: we label the N spins $\sigma_1, \dots, \sigma_N$ and think of them as cavity spins, and denote the M spins by τ_1, \dots, τ_M and think of them as the environment (the thermal bath) for the cavity.

The size M of the bath is now made large, at fixed N . An important effect of taking this limit is that the fields acting on the cavity spins are dominated by their interactions with the bath rather than their interactions with each other. In the limit $M \rightarrow \infty$ the cavity spins then become effectively non-interacting with each other and live in uncorrelated fields whose statistics are governed by

those of the bath. We will now detail this important motivation for the ROST approach.

We define and decompose the Hamiltonian $H_{M+N}(\sigma, \tau)$ of the overall $M+N$ -spin system as

$$H_{M+N}(\sigma, \tau) = -\frac{1}{\sqrt{M+N}} \sum_{1 \leq k < l \leq M} J_{kl} \tau_k \tau_l \quad (1)$$

$$- \frac{1}{\sqrt{M+N}} \sum_{1 \leq k \leq M, 1 \leq i \leq N} \tilde{J}_{ki} \tau_k \sigma_i - \frac{1}{\sqrt{M+N}} \sum_{1 \leq i < j \leq N} \hat{J}_{ij} \sigma_i \sigma_j,$$

where the relevant interaction variables J_{kl} , \tilde{J}_{ki} and \hat{J}_{ij} are all independent standard Gaussian random variables. Now call the first term in the Hamiltonian (1) $H_{M+N}(\tau)$ and write the second one as $\sum_{1 \leq i \leq N} \tilde{h}_i(\tau) \sigma_i$ with $\tilde{h}_i(\tau) = -\frac{1}{\sqrt{M+N}} \sum_{1 \leq k \leq M} \tilde{J}_{ki} \tau_k$ which for large M becomes

$$\tilde{h}_i(\tau) = -\frac{1}{\sqrt{M}} \sum_{1 \leq k \leq M} \tilde{J}_{ki} \tau_k. \quad (2)$$

The third contribution in (1), which has the interactions among the cavity spins, is a sum over only N^2 terms. This is at most $O(N^2/\sqrt{(M+N)})$ and goes to zero for $M \rightarrow \infty$ as anticipated.

Similarly the Hamiltonian governing the M -spin bath can be written as

$$H_M(\tau) = H_{M+N}(\tau) - \left(\frac{1}{M} - \frac{1}{M+N}\right)^{1/2} \sum_{1 \leq k < l \leq M} \hat{J}_{kl} \tau_k \tau_l$$

Here the random interactions \hat{J}_{kl} are independent from all others (and not related to the \hat{J}_{ij} above, the latter referring to σ - σ interactions). The above decomposition of $H_M(\tau)$ can be understood by noting that it gives for each bond strength a variance of $1/(M+N) + 1/M - 1/(M+N) = 1/M$ as it should be. For large M we can then write

$$H_M(\tau) = H_{M+N}(\tau) + \sqrt{\frac{N}{2}} \hat{H}(\tau), \quad \hat{H}(\tau) = -\frac{\sqrt{2}}{M} \sum_{1 \leq k < l \leq M} \hat{J}_{kl} \tau_k \tau_l.$$

Putting both together gives for large M at any fixed N for the difference between the log partition functions of the $M+N$ and M -spin systems

$$\mathbb{E} \ln \frac{Z_{M+N}(\beta)}{Z_M(\beta)} = \mathbb{E} \ln \frac{\sum_{\sigma, \tau} e^{-\beta H_{M+N}(\tau) - \beta \sum_i \tilde{h}_i(\tau) \sigma_i}}{\sum_{\tau} e^{-\beta H_{M+N}(\tau) - \beta (N/2)^{1/2} \hat{H}(\tau)}}, \quad (3)$$

where \mathbb{E} represents the disorder average over the couplings.

Now let us call

$$\xi(\tau) = e^{-\beta H_{M+N}(\tau)}, \quad (4)$$

and symmetrize $\hat{H}(\tau)$ w.r.t. the ordering of k and l by defining i.i.d. unit Gaussian random variables J_{kl} for *all* pairs (k, l) such that

$$\hat{H}(\tau) = -\frac{1}{M} \sum_{1 \leq k \neq l \leq M} J_{kl} \tau_k \tau_l. \quad (5)$$

We can also add the diagonal terms and modify $\hat{H}(\tau)$ to

$$\hat{H}(\tau) = -\frac{1}{M} \sum_{1 \leq k, l \leq M} \hat{J}_{kl} \tau_k \tau_l. \quad (6)$$

The resulting extra Gaussian random contribution is τ -independent and so pulls through the sum over all τ and the logarithm in (3) to appear linearly in the expectation over disorder, where it then vanishes. Calling $P(\beta)$ the thermodynamic pressure, defined in terms of free energy density $f(\beta)$ as $P(\beta) = -\beta f(\beta)$ and using $\mathbb{E} \ln Z_{M+N}(\beta) = (M+N)P(\beta)$ for large M gives finally

$$P(\beta) = \frac{1}{N} \mathbb{E} \ln \frac{Z_{M+N}(\beta)}{Z_M(\beta)} = \frac{1}{N} \mathbb{E} \ln \frac{\sum_{\sigma, \tau} \xi(\tau) e^{-\beta \sum_i \tilde{h}_i(\tau) \sigma_i}}{\sum_{\tau} \xi(\tau) e^{-\beta (N/2)^{1/2} \hat{H}(\tau)}}. \quad (7)$$

The $\tilde{h}_i(\tau)$ and $\hat{H}(\tau)$ are all zero mean Gaussian random variables. The two families of variables are uncorrelated with each other, while within the families the covariances are

$$\mathbb{E} \tilde{h}_i(\tau) \tilde{h}_j(\tau') = \delta_{ij} \frac{1}{M} \sum_{1 \leq k \leq M} \tau_k \tau'_k = \delta_{ij} q(\tau, \tau') \quad (8)$$

$$\mathbb{E} \hat{H}(\tau) \hat{H}(\tau') = \frac{1}{M^2} \sum_{1 \leq k, l \leq M} \tau_k \tau_l \tau'_k \tau'_l = q^2(\tau, \tau'). \quad (9)$$

A Random Overlap Structure or ROST is a generalization of the above structure which allows one to describe more generally (for example in terms of a Parisi ultrametric tree) the states τ of the bath for the cavity spins σ . Similarly the overlaps between these states are left unspecified, and hence denoted with a tilde, as are the weights $\xi(\tau)$.

Let us then start by defining a *Random Overlap Structure* \mathcal{R} as a triple (Σ, \tilde{q}, ξ) where

- Σ is a discrete space;
- $\xi : \Sigma \rightarrow \mathbb{R}_+$ is a system of random weights;
- $\tilde{q} : \Sigma^2 \rightarrow [-1, 1]$ is a symmetric *Overlap Kernel*, with $\tilde{q}(\tau, \tau) = 1$.

Now consider two families of independent centred Gaussian random variables \tilde{h}_i and \hat{H} , defined on $\Sigma \ni \tau$, such that there are N variables $\tilde{h}_i(\tau)$, for each τ and

$$\mathbb{E}(\tilde{h}_i(\tau) \tilde{h}_j(\tau')) = \tilde{q}(\tau, \tau') \delta_{ij}, \quad \mathbb{E}(\hat{H}(\tau) \hat{H}(\tau')) = \tilde{q}^2(\tau, \tau'). \quad (10)$$

Then the *Generalized Trial Pressure* can be written as

$$G_N(\mathcal{R}) = \frac{1}{N} \mathbb{E} \ln \frac{\sum_{\sigma, \tau} \xi(\tau) \exp(-\beta \sum_{i=1}^N \tilde{h}_i(\tau) \sigma_i)}{\sum_{\tau} \xi(\tau) \exp(-\beta (N/2)^{1/2} \hat{H}(\tau))}. \quad (11)$$

In the following two subsections we outline the properties of the ROST defined above, following the presentation in [3],[24]: we state the required theorems concerning the optimality of the ROST and we introduce the Boltzman ROST, referring the interested reader to the original papers for the proofs.

2.2 The Boltzmann ROST

Equations (2,4,6) define the Boltzmann ROST [24], with one exception. In the Hamiltonian $H_{M+N}(\tau)$ that defines the weights $\xi(\tau)$, and is given by the first term on the r.h.s. of (1), the normalizing prefactor $1/\sqrt{M+N}$ is replaced by $1/\sqrt{M}$. This is equivalent to replacing this Hamiltonian by $H_M(\tau)$, which represents only the interactions within the bath.

On a superficial level, this change is necessary to comply with the general definition of a ROST: the weights $\xi(\tau)$ must not depend on N . At first sight it looks dangerous, however for the dominant states τ , $H_{M+N}(\tau)$ is $O(M)$, and the change of the prefactor by $\sqrt{(M+N)/M} = 1 + N/(2M)$ modifies the Hamiltonian by a term of $O(N)$ that remains non-negligible even for large M . Fortunately, the prefactor shift can alternatively be regarded as a slight temperature shift to $\beta^* = \beta\sqrt{(M+N)/M}$. Evaluating (7) at this temperature and using $\beta^* H_{M+N}(\tau) = \beta H_M(\tau)$ gives

$$P(\beta\sqrt{M+N/M}) = \frac{1}{N} \mathbb{E} \ln \frac{\sum_{\sigma, \tau} e^{-\beta H_M(\tau)} e^{-\beta\sqrt{(M+N)/M} \sum_i \tilde{h}_i(\tau) \sigma_i}}{\sum_{\tau} e^{-\beta H_M(\tau)} e^{-\beta\sqrt{M+N/M} (N/2)^{1/2} \hat{H}(\tau)}}. \quad (12)$$

The two exponents on the r.h.s. where the factor $\sqrt{(M+N)/M}$ now appears do not grow with M so in them one can replace $\sqrt{(M+N)/M}$ by 1 for large M . Similarly, as the pressure is a continuous function of the inverse temperature, the l.h.s. tends to $P(\beta)$ for large M . This shows that (7) remains correct if we define the weights as $\xi(\tau) = \exp(-\beta H_M(\tau))$, as claimed.

If we now call $\mathcal{R}_B(M)$ the Boltzmann ROST we have just defined, one can prove the following

Theorem 2.1 (Reversed Bound).

$$-\beta f(\beta) \geq \lim_{N \rightarrow \infty} \liminf_{M \rightarrow \infty} G_N(\mathcal{R}_B(M))$$

The idea of the proof, which we do not elaborate here, is to compare $G_N(\mathcal{R}_B(M))$ with

$$\frac{1}{N} \mathbb{E} \ln \frac{Z_{N+M}(\beta)}{Z_M(\beta)}.$$

The following theorem states that the generalized trial pressure provides an upper bound on the SK model pressure, i.e. a lower bound for the free energy.

Theorem 2.2 (Generalized Bound).

$$-\beta f_N(\beta) \leq \inf_{\mathcal{R}} G_N(\mathcal{R}).$$

From the two previous theorems one gets immediately the following

Theorem 2.3 (Extended Variational Principle).

$$-\beta f(\beta) = \lim_{N \rightarrow \infty} \inf_{\mathcal{R}} G_N(\mathcal{R}).$$

The theorem implies that it is sufficient to limit our trial functions to those depending on trial overlaps, like those in the ROST space, and expressed as

the difference between a cavity term and an internal energy part, as in the numerator and denominator of the generalized trial pressure G_N .

We will first decompose the generalized trial pressure $G(\mathcal{R})$ from (11), evaluated for the Boltzmann ROST $\mathcal{R}_B(M)$, into two parts. Seeing as the weights $\xi(\tau) = \exp(-\beta H_M(\tau))$ of the bath configurations τ are simply Boltzmann weights, we can introduce the notation

$$\omega(F) = \frac{\sum_{\tau} F(\tau) \exp(-\beta H_M(\tau))}{\sum_{\tau} \exp(-\beta H_M(\tau))}$$

for the bath Boltzmann state. The generalized trial pressure can then be written as

$$G_N(\mathcal{R}_B(M)) = \frac{1}{N} \mathbb{E} \ln \omega \left(\sum_{\sigma} \exp(-\beta \sum_{i=1}^N \tilde{h}_i \sigma_i) \right) - \frac{1}{N} \mathbb{E} \ln \omega \left(\exp(-\beta \sqrt{\frac{N}{2}} \hat{H}) \right). \quad (13)$$

We will see that this decomposition mirrors exactly the one in [6], and therefore call the second term the “internal energy term” and the first the “entropy term”. These names are not quite precise but act as convenient shorthands. If $e(\beta)$ is the SK internal energy, then the internal energy term is in fact $-(1/2)\beta e(\beta)$, while the entropy term contains in addition to the entropy a contribution of $-(3/2)\beta e(\beta)$.

Let us now consider the internal energy term of the Boltzmann ROST generalized trial pressure, but with the β in the exponent generalized to some β' that can be different from the inverse temperature defining the bath Boltzmann state. Thanks to the stochastic stability of the Gibbs measure [14], one can show that this affects the result only through a prefactor [7, 8], as stated in the following

Theorem 2.4 (The energy expression). *For $M \rightarrow \infty$,*

$$\frac{1}{N} \mathbb{E} \ln \omega \exp \left(-\beta' \sqrt{\frac{N}{2}} \hat{H}(\tau) \right) = \frac{\beta'^2}{4} (1 - \langle \tilde{q}_{12}^2 \rangle). \quad (14)$$

On the r.h.s., $\langle \cdot \rangle = \mathbb{E} \Omega(\cdot)$ and Ω is the replicated bath Boltzmann state ω .

2.3 Introducing the RaMOST for the VB

The RaMOST plays a role analogous to that of the ROST for the SK when dealing with diluted system such as the Viana-Bray model (VB) [41]. A fundamental difference is that we now need another real parameter α to take into account the connectivity of the underlying random graph. We recall here that the VB model is a spin model defined on a random graph, with interactions only present on the edges of the graph.

As in the SK case, to motivate the RaMOST, consider a cavity of N spins $\sigma_1, \dots, \sigma_N$ in a “bath” of M spins τ_1, \dots, τ_M , with $M \gg N$. The aim is to obtain the pressure by considering the free energy increment when going from M to $M+N$ spins, $P(\beta, \bar{\alpha}) = (1/N) \mathbb{E} [\ln Z_{M+N}(\beta, \bar{\alpha}) - Z_N(\beta, \bar{\alpha})]$. Based on experience with the SK model, where we had to allow for a slight temperature shift to construct the Boltzmann ROST, we allowed here for a shifted connectivity $\bar{\alpha}$

which should approach our desired connectivity α when $M/N \rightarrow \infty$. To write the partition function of the $(M+N)$ -spin system, decompose the Hamiltonian as in [8]

$$-H_{M+N}(\sigma, \tau, \bar{\alpha}) = \sum_{\nu=1}^{P_{\zeta}} J_{\nu} \tau_{i_{\nu}} \tau_{l_{\nu}} + \sum_{\nu=1}^{P_{\tilde{\zeta}}} \tilde{J}_{\nu} \tau_{i_{\nu}} \sigma_{\tilde{j}_{\nu}} + \sum_{\nu=1}^{P_{\zeta'}} J'_{\nu} \sigma_{j'_{\nu}} \sigma_{k'_{\nu}} . \quad (15)$$

All the J -variables here are i.i.d. interaction strengths, distributed symmetrically about zero (e.g. binary values ± 1 as used in the following, or zero mean Gaussian variables; the precise choice should be unimportant as mean field spin glasses are thought to display universality [12]). The spin indices i_{ν} etc are uniformly distributed across $\{1, \dots, M\}$ or $\{1, \dots, N\}$ as appropriate. The upper summation limits P_{ζ} etc. are Poisson random variables with mean number of bonds in each “sector” given by

$$\zeta = \bar{\alpha} \frac{M^2}{M+N} , \quad \tilde{\zeta} = \bar{\alpha} \frac{2MN}{M+N} , \quad \zeta' = \bar{\alpha} \frac{N^2}{M+N} .$$

For example, the total mean number of bonds is by definition $(M+N)\bar{\alpha}$; there are M^2 “ $\tau\tau$ ” spin pairs out of a total of $(M+N)^2$ and hence $\zeta = \bar{\alpha}(M+N)[M^2/(M+N)^2] = \bar{\alpha}M^2/(M+N)$. To make the $\tau\tau$ part of the Hamiltonian equivalent to an M -spin Hamiltonian with connectivity α , we need $\zeta = \alpha M$ and thus

$$\bar{\alpha} = \alpha \frac{M+N}{M} . \quad (16)$$

As $\zeta' \rightarrow 0$ for $M \rightarrow \infty$ at fixed N , this term in H_{M+N} can be discarded with probability one: as in the SK case, making the bath large enough allows us to neglect interactions of the cavity spins. Summarizing so far, we have for large M

$$\begin{aligned} Z_{M+N}(\beta, \bar{\alpha}) &= \sum_{\sigma, \tau} \exp \left(\beta \sum_{\nu=1}^{P_{\alpha M}} J_{\nu} \tau_{i_{\nu}} \tau_{l_{\nu}} + \beta \sum_{\nu=1}^{P_{2\alpha N}} \tilde{J}_{\nu} \tau_{i_{\nu}} \sigma_{\tilde{j}_{\nu}} \right) \\ &= \sum_{\sigma, \tau} \exp \left(-\beta H_M(\tau, \alpha) - \beta \tilde{H}(\sigma, \tau, \alpha) \right) \end{aligned} \quad (17)$$

where

$$\tilde{H}(\sigma, \tau, \alpha) = - \sum_{\nu=1}^{P_{2\alpha N}} \tilde{J}_{\nu} \tau_{i_{\nu}} \sigma_{\tilde{j}_{\nu}} = \sum_{j=1}^N \tilde{h}_j(\tau) \sigma_j . \quad (18)$$

Here $\tilde{h}_j(\tau)$ is the cavity field acting on σ_j defined by

$$\tilde{h}_j(\tau) = - \sum_{\nu=1}^{P_{2\alpha}} \tilde{J}_{\nu}^j \tau_{i_{\nu}^j} ,$$

and the index j of \tilde{J}_{ν}^j and i_{ν}^j indicates independent copies of the corresponding random variables. The first form of \tilde{H} given in (18) is more useful for our calculations, while the second one emphasizes the physics: as in the SK model, each cavity spin σ_j experiences a cavity field arising from its interaction with the bath. In the VB case, this field is due to a Poisson-distributed number (with mean 2α) of interactions with randomly chosen spins $\tau_{i_{\nu}^j}$ from the bath.

To write the partition function of the M -spin system with connectivity $\bar{\alpha}$ in a similar form, we write its number of bonds as $P_{\bar{\alpha}M} = P_{\alpha M} + P_{\alpha N}$:

$$\begin{aligned} Z_M(\beta, \bar{\alpha}) &= \sum_{\tau} \exp \left(\beta \sum_{\nu=1}^{P_{\alpha M}} J_{\nu} \tau_{i_{\nu}} \tau_{l_{\nu}} + \beta \sum_{\nu=1}^{P_{\alpha N}} \hat{J}_{\nu} \tau_{\hat{i}_{\nu}} \tau_{\hat{l}_{\nu}} \right) \\ &= \sum_{\tau} \exp \left(-\beta H_M(\tau, \alpha) - \beta \hat{H}(\tau, \alpha) \right) \end{aligned} \quad (19)$$

where

$$\hat{H}(\tau, \alpha) = - \sum_{\nu=1}^{P_{\alpha N}} \hat{J}_{\nu} \tau_{\hat{i}_{\nu}} \tau_{\hat{l}_{\nu}}. \quad (20)$$

Defining Boltzmann weights $\xi(\tau) = \exp(-\beta H_M(\tau, \alpha))$, we can then write for large M

$$P(\beta, \alpha) = \frac{1}{N} \mathbb{E} \ln \frac{Z_{M+N}(\beta, \bar{\alpha})}{Z_M(\beta, \bar{\alpha})} = \frac{1}{N} \mathbb{E} \ln \frac{\sum_{\sigma, \tau} \xi(\tau) e^{-\beta \hat{H}(\sigma, \tau, \alpha)}}{\sum_{\tau} \xi(\tau) e^{-\beta \hat{H}(\tau, \alpha)}}. \quad (21)$$

By default the connectivity at which the pressure is found in this way is $\bar{\alpha}$, but we have already exploited the fact that for large M this tends to α .

In order to think at the above representation of the pressure (which so far we have mainly tried to motivate, without being rigorous) as the generalized trial pressure of a Random Multi-Overlap Structure (RaMOST), we need to show that the statistics of \tilde{H} and \hat{H} can be expressed in terms of multi-overlaps of the bath states τ . To see this, note that the definitions (18) and (20) of both quantities can be written in terms of sums over i.i.d. variables:

$$\hat{H}(\tau, \alpha) = \sum_{\nu=1}^{P_{\alpha N}} \hat{h}^{\nu}(\tau), \quad (22)$$

$$\tilde{H}(\sigma, \tau, \alpha) = \sum_{\nu=1}^{P_{2\alpha N}} \tilde{h}^{\nu}(\tau) \sigma_{\tilde{j}_{\nu}}, \quad (23)$$

where $\hat{h}^{\nu}(\tau)$ and $\tilde{h}^{\nu}(\tau)$ are independent copies of random variables $\hat{h}(\tau)$ and $\tilde{h}(\tau)$. The latter form two independent families of random variables indexed by τ , whose probability distributions have even moments

$$\mathbb{E}[\hat{h}(\tau^1) \cdots \hat{h}(\tau^{2n})] = (\mathbb{E} J^{2n}) \tilde{q}_{2n}^2(\tau^1, \dots, \tau^{2n}), \quad (24)$$

$$\mathbb{E}[\tilde{h}(\tau^1) \cdots \tilde{h}(\tau^{2n})] = (\mathbb{E} J^{2n}) \tilde{q}_{2n}(\tau^1, \dots, \tau^{2n}), \quad (25)$$

while the odd moments vanish. These properties follow from the fact that in our construction so far $\hat{h}(\tau) = \hat{J} \tau_{\hat{i}} \tau_{\hat{j}}$ and $\tilde{h}(\tau) = \tilde{J} \tau_{\hat{i}}$. The bath multi-overlaps occurring above are then

$$\tilde{q}_{2n}(\tau^1, \dots, \tau^{2n}) = \frac{1}{M} \sum_{1 \leq i \leq M} \tau_i^1 \cdots \tau_i^{2n}. \quad (26)$$

We can now generalize and allow generic ways of specifying the states τ of the bath and their multi-overlaps \tilde{q}_{2n} . At this point it is clear that we have outlined essentially the same setting as the one we used for the SK model, and the previous remarks allow us to introduce the *Random Multi-Overlap Structure* \mathcal{R} as a triple $(\Sigma, \{\tilde{q}_{2n}\}, \xi)$ where

- Σ is a discrete space;
- $\xi : \Sigma \rightarrow \mathbb{R}_+$ is a system of random weights;
- $\tilde{q}_{2n} : \Sigma^{2n} \rightarrow [-1, 1], n \in \mathbb{N}$ is a *Multi-Overlap Kernel*. This needs to be such that (24, 25) define valid random variables $\hat{h}(\tau)$ and $\tilde{h}(\tau)$, and in particular each \tilde{q}_{2n} must be symmetric in its arguments. The multi-overlap kernels for different n must also be linked by the following *reduction property*: $\tilde{q}_{2n+2}(\tau, \tau, \dots) = \tilde{q}_{2n}(\dots)$ for $n \in \mathbb{N}$ and $\tilde{q}_2(\tau, \tau) = 1$.

The generalized trial pressure for such a RaMOST is then defined as

$$G_N(\mathcal{R}) = \frac{\sum_{\sigma, \tau} \xi(\tau) e^{-\beta \tilde{H}(\sigma, \tau, \alpha)}}{\sum_{\tau} \xi(\tau) e^{-\beta \hat{H}(\tau, \alpha)}}, \quad (27)$$

where the statistics of the random variables $\tilde{H}(\sigma, \tau, \alpha)$ and $\hat{H}(\tau, \alpha)$ are as defined by (22–25).

Note that the factorization of (24, 25) implies that $\hat{h}^\nu(\tau)$ and $\tilde{h}^\nu(\tau)$ can be written as $\hat{J}\hat{\epsilon}^\nu(\tau)$ and $\tilde{J}\tilde{\epsilon}^\nu(\tau)$. The reduction property of the kernel then further shows that $\hat{\epsilon}(\tau)$ and $\tilde{\epsilon}(\tau)$ are binary (± 1), because all their even moments are unity ($\mathbb{E}[\hat{\epsilon}^{2n}(\tau)] = \tilde{q}_{2n}^2(\tau, \dots, \tau) = 1$ and similarly for $\tilde{\epsilon}$) while the odd ones vanish.

We will call the RaMOST introduced above, where $\Sigma = \{-1, +1\}^M$, $\xi(\tau) = \exp(-\beta H_M(\tau, \alpha))$ are Boltzmann weights and the multi-overlaps are as in (26), the *Boltzmann RaMOST* $\mathcal{R}_B(M)$. The reduction property is then entirely natural: even numbers of replicas cancel to give e.g. $q_4(\tau^1, \tau^1, \tau^1, \tau^2) = q_2(\tau^1, \tau^2)$ and $q_4(\tau^1, \tau^1, \tau^2, \tau^2) = q_2(\tau^2, \tau^2) = 1$.

The generality of the RaMOST allows one, on the other hand, to take Σ (which is not necessarily $\{-1, +1\}^M$) as the set of indices τ of the weights $\xi(\tau)$ constructed by means of Random Probability Cascades of Poisson-Dirichlet processes (see e.g. Ref. [31]). These cascades give rise to nested chains of expectations of Parisi type, and reproduce the Parisi Replica Symmetry Breaking theory if one interpolates according to the iterative approach of Refs. [21, 31].

2.4 The Boltzmann RaMOST

Now to acquire familiarity with the RaMOST framework we state a package of theorems mirroring the Aizenman, Sims and Starr theory for the SK free energy [18].

Consider for $t \in [0, 1]$ and a given RaMOST \mathcal{R} the following *interpolating Hamiltonian*

$$H(\sigma, \tau, t) = H_N(\sigma, t\alpha) + \hat{H}(\tau, t\alpha) + \tilde{H}(\sigma, \tau, (1-t)\alpha)$$

and using the RaMOST weights $\xi(\tau)$ define

$$g_N(t) = \frac{1}{N} \mathbb{E} \ln \frac{\sum_{\sigma, \tau} \xi(\tau) \exp(-\beta H(\sigma, \tau, t))}{\sum_{\tau} \xi(\tau) \exp(-\beta \hat{H}(\tau, \alpha))}. \quad (28)$$

Clearly then

$$\begin{aligned} g_N(0) &= G_N(\mathcal{R}), \\ g_N(1) &= -\beta f_N(\beta, \alpha). \end{aligned}$$

Within this construction the following results easily follow [3, 7].

Theorem 2.5 (Generalized Bound).

$$P_N(\beta, \alpha) \equiv -\beta f_N(\beta, \alpha) \leq \inf_{\mathcal{R}} G_N(\mathcal{R}) .$$

Theorem 2.6 (Reversed Bound).

$$P(\beta, \alpha) \equiv -\beta f(\beta, \alpha) \geq \inf_{\mathcal{R}} G_N(\mathcal{R}) .$$

Theorem 2.7 (Extended Variational Principle).

$$P(\beta, \alpha) \equiv -\beta f(\beta, \alpha) = \lim_{N \rightarrow \infty} \inf_{\mathcal{R}} G_N(\mathcal{R}) .$$

Theorem 2.8 (The energy expression). *Let ω , $\langle \cdot \rangle$ be the usual Boltzmann-Gibbs and quenched Boltzmann-Gibbs expectations at inverse temperature β , associated with the Hamiltonian $H_N(\sigma, \alpha)$. Then*

$$\lim_{N \rightarrow \infty} \mathbb{E} \ln \omega \exp \left(\beta' \sum_{\nu=1}^{P_{\bar{\alpha}}} J'_{\nu} \sigma_{i'_{\nu}} \sigma_{j'_{\nu}} \right) = \bar{\alpha} \sum_{n=1}^{\infty} \frac{1}{2n} \tanh^{2n}(\beta') (1 - \langle q_{2n}^2 \rangle) , \quad (29)$$

where the random variables $\{J'_{\nu}\}$, $\{i'_{\nu}\}$ and $\{j'_{\nu}\}$ are independent copies of the analogous random variables appearing in the Hamiltonian in ω , and $P_{\bar{\alpha}}$ is a Poisson random variable with mean $\bar{\alpha}$. On the r.h.s. the quenched Boltzmann-Gibbs expectation is of the square of the multi-overlap q_{2n} of n replicas of the system, defined as in (26).

3 The general expansion of the “energy” term

We want to expand the “energy” term

$$e = \mathbb{E} \ln \Omega(\exp[-\beta' \hat{H}(\tau)]) \quad (30)$$

in β' in order to compare this expansion with the r.h.s. of Eq. (14) for the SK model and Eq. (29) for the VB model. We use Ω to denote the Boltzmann measure of τ (whose form will not matter), both for a single replica and later also for the corresponding replicated measure.

In this section, we find a suitably general form of the expansion of e , which does not rely on the specific form of $\hat{H}(\tau)$.

We expand first the exponential

$$\Omega(\exp[-\beta' \hat{H}(\tau)]) = 1 + \sum_{n \geq 1} \frac{(-\beta')^n}{n!} \Omega(\hat{H}^n(\tau)) \quad (31)$$

and then the log to get

$$e = \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} \sum_{n_1 \dots n_m \geq 1} \frac{(-\beta')^{n_1 + \dots + n_m}}{n_1! \dots n_m!} \mathbb{E} \left[\Omega(\hat{H}^{n_1}(\tau)) \dots \Omega(\hat{H}^{n_m}(\tau)) \right] . \quad (32)$$

The expectation appearing here can be rewritten as

$$\mathbb{E} \Omega \left[\hat{H}(\tau^1) \dots \hat{H}(\tau^1) \times \dots \times \hat{H}(\tau^n) \dots \hat{H}(\tau^n) \right] \quad (33)$$

where replica τ^1 appears n_1 times, τ^2 appears n_2 times and so on. Now group terms according to $n = n_1 + \dots + n_m$, bearing in mind that $n \geq m$, and use the shorthand $\langle \dots \rangle = \mathbb{E}\Omega(\dots)$:

$$e = \sum_{n \geq 1} \frac{(-\beta')^n}{n!} \sum_{m=1}^n \frac{(-1)^{m-1}}{m} \sum_{n_1 \dots n_m \geq 1, n_1 + \dots + n_m = n} \frac{n!}{n_1! \dots n_m!} \times \langle \hat{H}(\tau^1) \dots \hat{H}(\tau^1) \times \dots \times \hat{H}(\tau^n) \dots \hat{H}(\tau^n) \rangle \quad (34)$$

The combinatorial factor $n!/(n_1! \dots n_m!)$ just gives the number of permutations of the replica indices inside the $\langle \dots \rangle$, so one can write equivalently

$$e = \sum_{n \geq 1} \frac{(-\beta')^n}{n!} \sum_{m=1}^n \frac{(-1)^{m-1}}{m} \sum'_{1 \leq a_1, \dots, a_n \leq m} \langle \hat{H}(\tau^{a_1}) \dots \hat{H}(\tau^{a_n}) \rangle. \quad (35)$$

The prime on the last sum corresponds to the constraints $n_1, \dots, n_m \geq 1$: only terms in which each of the m replicas appears at least once are to be included. In other words, as an identity between sets (where multiple occurrences count as one) we must have $\{a_1, \dots, a_n\} = \{1, \dots, m\}$.

We now exploit permutation symmetry of replicas to modify the sum over a_1, \dots, a_n by expanding its summation range. This looks more complicated initially but will pay dividends shortly by producing unrestricted sums. From permutation symmetry, our expansion is unchanged if we let a_1, \dots, a_n take values in some general subset T of $\{1, \dots, n\}$, of size $|T| = m$. The constraint on the summation would then be $|\{a_1, \dots, a_n\}| = T$. We can now sum over all $n!/m!(n-m)!$ possible choices of T and divide by this factor. The possible assignments of a_1, \dots, a_n that result from this summation over T are clearly all distinct, and together give precisely all the assignments of a_1, \dots, a_n — in the now *expanded* range $1, \dots, n$ — for which the set $\{a_1, \dots, a_n\}$ has exactly m elements. If we denote this constraint with a superscript (m) on the sum, we have

$$e = \sum_{n \geq 1} \frac{(-\beta')^n}{n!} \sum_{m=1}^n \frac{(-1)^{m-1}}{m} \frac{m!(n-m)!}{n!} \sum_{1 \leq a_1, \dots, a_n \leq n}^{(m)} \langle \hat{H}(\tau^{a_1}) \dots \hat{H}(\tau^{a_n}) \rangle, \quad (36)$$

but now the sum over m together with the constrained sum over the a_1, \dots, a_n just yields an unconstrained sum. We just need to bear in mind that the coefficient is m -dependent, i.e.

$$e = \sum_{n \geq 1} \frac{(-\beta')^n}{n!^2} \sum_{1 \leq a_1, \dots, a_n \leq n} (-1)^{m-1} (m-1)!(n-m)! \langle \hat{H}(\tau^{a_1}) \dots \hat{H}(\tau^{a_n}) \rangle, \quad (37)$$

where now $m = |\{a_1, \dots, a_n\}|$ is a function of the a_1, \dots, a_n which counts the number of distinct members in the set of replica indices $\{a_1, \dots, a_n\}$. This is our desired general expansion, where now only unconstrained sums appear. In our cases of interest, the averages over \hat{H} vanish for odd n and we need only the even terms, i.e. after relabelling $n \rightarrow 2n$

$$e = \sum_{n \geq 1} \frac{\beta'^{2n}}{(2n)!^2} \sum_{1 \leq a_1, \dots, a_{2n} \leq 2n} (-1)^{m-1} (m-1)!(2n-m)! \langle \hat{H}(\tau^{a_1}) \dots \hat{H}(\tau^{a_{2n}}) \rangle. \quad (38)$$

4 Expansion in the SK model

For the SK model, we want to verify whether the identities we get are indeed of AC form [1], and to determine if and which subset of AC identities they produce. Note that the expansion parameter in e is in principle $\beta' \sqrt{N/2}$ instead of β' . We keep β' for now; the original version can be retrieved at any point trivially by reinstating $\beta' \rightarrow \beta' \sqrt{N/2}$. The perturbation Hamiltonian in the exponent is

$$\hat{H}(\tau) = -\frac{1}{M} \sum_{kl} \hat{J}_{kl} \tau_k \tau_l \quad (39)$$

with the \hat{J}_{kl} i.i.d. Gaussian random variables of zero mean and unit variance.

To simplify the expansion (38) we carry out part of the disorder average, over the \hat{J}_{kl} . Consider

$$\begin{aligned} \langle \hat{H}(\tau^{a_1}) \dots \hat{H}(\tau^{a_{2n}}) \rangle &= \\ &= M^{-2n} \sum_{k_1, l_1, \dots, k_{2n}, l_{2n}} \mathbb{E}(\hat{J}_{k_1 l_1} \dots \hat{J}_{k_{2n} l_{2n}}) \langle \tau_{k_1}^{a_1} \tau_{l_1}^{a_1} \dots \tau_{k_{2n}}^{a_{2n}} \tau_{l_{2n}}^{a_{2n}} \rangle, \end{aligned} \quad (40)$$

Wick's theorem gives a sum over pairings of the various indices $(k_1, l_1), (k_2, l_2)$ etc., or equivalently pairings of the replica indices a_1, \dots, a_{2n} . This can be written as a sum over permutations π of $2n$ elements if we bear in mind that we then overcount each pairing $2^n n!$ times:

$$\begin{aligned} \langle \hat{H}(\tau^{a_1}) \dots \hat{H}(\tau^{a_{2n}}) \rangle &= \\ &= \frac{1}{2^n n!} M^{-2n} \sum_{\pi} \sum_{k_1, l_1, \dots, k_n, l_n} \langle \tau_{k_1}^{a_{\pi(1)}} \tau_{l_1}^{a_{\pi(1)}} \tau_{k_1}^{a_{\pi(2)}} \tau_{l_1}^{a_{\pi(2)}} \dots \\ &\quad \dots \tau_{k_n}^{a_{\pi(2n-1)}} \tau_{l_n}^{a_{\pi(2n-1)}} \tau_{k_n}^{a_{\pi(2n)}} \tau_{l_n}^{a_{\pi(2n)}} \rangle = \\ &= \frac{1}{2^n n!} \sum_{\pi} \langle q_{a_{\pi(1)} a_{\pi(2)}}^2 \dots q_{a_{\pi(2n-1)} a_{\pi(2n)}}^2 \rangle. \end{aligned} \quad (41)$$

Now we insert this into the general expansion (38). Because the summation over a_1, \dots, a_{2n} is symmetric, each permutation π gives the same contribution and the sum over π therefore just yields a factor $(2n)!$ so that

$$e^{\text{SK}} = - \sum_{n \geq 1} \beta'^{2n} \langle E_n^{\text{SK}} \rangle \quad (42)$$

with

$$E_n^{\text{SK}} = \frac{1}{2^n (2n)! n!} \sum_{1 \leq a_1, \dots, a_{2n} \leq 2n} (-1)^m (m-1)! (2n-m)! \langle q_{a_1 a_2}^2 \dots q_{a_{2n-1} a_{2n}}^2 \rangle. \quad (43)$$

This form of the result is beginning to look useful, but there is the complication that, when e.g. $a_1 = a_2$, $q_{a_1 a_2}^2 = 1$ so it looks like various orders of q are mixed. We therefore next show that the sum can be restricted to the terms where $a_1 \neq a_2$, $a_3 \neq a_4$ etc. To see this, insert into the sum appearing in the

expression for E_n^{SK} a factor

$$\begin{aligned}
1 &= [(1 - \delta_{a_1 a_2}) + \delta_{a_1 a_2}] \cdots [(1 - \delta_{a_{2n-1} a_{2n}}) + \delta_{a_{2n-1} a_{2n}}] \\
&= (1 - \delta_{a_1 a_2}) \cdots (1 - \delta_{a_{2n-1} a_{2n}}) \\
&\quad + \delta_{a_1 a_2} (1 - \delta_{a_3 a_4}) \cdots (1 - \delta_{a_{2n-1} a_{2n}}) + \cdots \\
&\quad + \delta_{a_1 a_2} \delta_{a_3 a_4} (1 - \delta_{a_5 a_6}) \cdots (1 - \delta_{a_{2n-1} a_{2n}}) + \cdots \\
&\quad + \cdots \\
&\quad + \delta_{a_1 a_2} \cdots \delta_{a_{2n-1} a_{2n}}.
\end{aligned} \tag{44}$$

We want to show that all the terms containing at least one factor δ_{ab} (i.e. all except those in the first line) vanish once summed over. This is easy to see. Consider without loss of generality $\delta_{a_{2n-1} a_{2n}}$ and fix all other summation indices. Call the set of these indices $S = \{a_1, \dots, a_{2n-2}\}$ and its size $s = |S|$. Now do the summation over $a_{2n-1} = a_{2n}$ in (42), noting that the average $\langle \dots \rangle$ is independent of which value a_{2n-1} takes (since $q_{a_{2n-1} a_{2n-1}}^2 = 1$). Only m varies: either $m = s$ if $a_{2n-1} \in S$, or $m = s + 1$ if $a_{2n-1} \notin S$. There are s values of a_{2n-1} where the first case occurs, and $2n - s$ values for the second case. Thus

$$\begin{aligned}
&\sum_{1 \leq a_{2n-1} = a_{2n} \leq 2n} (-1)^m (m-1)! (2n-m)! \langle \dots \rangle = \\
&s(-1)^s (s-1)! (2n-s)! + (2n-s)(-1)^{s+1} s! (2n-s-1)! = 0,
\end{aligned}$$

and any summation over the remaining indices a_1, \dots, a_{2n-2} (whether or not they contain further pairs of identical indices) of course then also gives a vanishing result. There is one exception to this argument: if $n = 1$ then $s = 0$, and $m = 1$ whatever the value of $a_1 = a_2$; here no cancelation can occur (mathematically, the breakdown of the argument is reflected in the appearance of the divergent factor $(s-1)! = (-1)!$ above). The $n = 1$ term is therefore separated off explicitly below.

We have now shown that in (42) we need to consider only distinct summation indices within each pair, i.e. $a_1 \neq a_2$ etc. We can further order the indices within each pair and then need to multiply by a factor 2^n to compensate, giving (the δ_{n1} term accounts for the non-canceling term at $n = 1$)

$$\begin{aligned}
E_n^{\text{SK}} &= -\frac{1}{2} \delta_{n1} + \frac{1}{n! (2n)!} \sum_{\text{o.p.} \leq 2n} (-1)^m (m-1)! (2n-m)! q_{a_1 a_2}^2 \cdots q_{a_{2n-1} a_{2n}}^2 \tag{45} \\
&= -\frac{1}{2} \delta_{n1} + \frac{1}{n!} \sum_{2 \leq m \leq 2n} \sum_{\text{o.p.} \leq m}^{(m)} \frac{(-1)^m}{m} q_{a_1 a_2}^2 \cdots q_{a_{2n-1} a_{2n}}^2. \tag{46}
\end{aligned}$$

The subscript “o.p.” indicates a sum over ordered pairs, $1 \leq a_1 < a_2 \leq 2n$, $1 \leq a_3 < a_4 \leq 2n$ etc. In the last row of eq. (46) we have re-introduced a sum over m and a constrained sum over (ordered pairs of) replica indices with m distinct elements. We have then further compressed the summation range of the replica indices to $1, \dots, m$, multiplying by $(2n)!/[m!(2n-m)!]$ to compensate. Notice that in this last version, E_n^{SK} is no longer symmetric under permutation of the replicas. But as we only need E_n^{SK} under the expectation $\langle E_n^{\text{SK}} \rangle$, which is invariant to permutations of replicas, this does not matter. In the same manner, we will from now on treat expressions in terms of overlaps as identical as long

as they give the same expectation $\langle \dots \rangle$ (or, equivalently, without taking the expectation but after symmetrizing over all permutations of the replicas).

We can now state the identities that follow from the ROST energy expression (14). This contains only terms of order β'^2 on the r.h.s., so comparing with the expansion (42) shows that our desired identities are simply $\langle E_n \rangle = 0$ for $n \geq 2$. We next obtain a simple recursion for the E_n which shows that all these identities are of AC form as expected. Consider $n \geq 2$ and let S and s be as defined as above from the first $2n - 2$ summation indices, i.e. $S = \{a_1, \dots, a_{2n-2}\}$ and $s = |S|$. We now start from (45) and make a transformation similar to the one leading to (46) but only for these first $2n - 2$ summation indices. To this end we introduce a sum over $s = 2, \dots, 2n - 2$ and a corresponding sum over (ordered pairs of) a_1, \dots, a_{2n-2} constrained so that S has s distinct elements. Permutation symmetry tells us that we can compress the range of a_1, \dots, a_{2n-2} from $1, \dots, 2n$ to $1, \dots, s$, if we multiply by the number of subsets of size s , $(2n)!/[s!(2n-s)!]$. In this manner we get, if we abbreviate also $a = a_{2n-1}$ and $b = a_{2n}$,

$$E_n^{\text{SK}} = \frac{1}{n!} \sum_{s=2}^{2n-2} \frac{1}{s!(2n-s)!} \times \sum_{\text{o.p.} \leq s}^{(s)} \sum_{1 \leq a < b \leq 2n} (-1)^m (m-1)! (2n-m)! q_{a_1 a_2}^2 \cdots q_{a_{2n-3} a_{2n-2}}^2 q_{ab}^2. \quad (47)$$

Now we carry out the sum over a and b . The total number of different replica indices present, m , depends on the range in which a and b lie: if both are $\leq s$, we have $m = s$, if only one is $> s$, $m = s + 1$, and if both are $> s$, $m = s + 2$. Abbreviating $Q = q_{a_1 a_2}^2 \cdots q_{a_{2n-3} a_{2n-2}}^2$, the last sum from the previous equation becomes

$$\begin{aligned} & \sum_{1 \leq a < b \leq s} (-1)^s (s-1)! (2n-s)! Q q_{ab}^2 \\ & + \sum_{s < b \leq 2n} \sum_{1 \leq a \leq s} (-1)^{s+1} s! (2n-s-1)! Q q_{ab}^2 \\ & + \sum_{s < a < b \leq 2n} (-1)^{s+2} (s+1)! (2n-s-2)! Q q_{ab}^2 \end{aligned} \quad (48)$$

Exploiting permutation symmetry among replica indices in the range $s+1, \dots, 2n$ — given that Q is a function only of replicas $1, \dots, s$ — and gathering prefactors simplifies this further to

$$(-1)^s (s-1)! (2n-s)! Q \left[\sum_{1 \leq a < b \leq s} q_{ab}^2 - s \sum_{1 \leq a \leq s} q_{a, s+1}^2 + \frac{s(s+1)}{2} q_{s+1, s+2}^2 \right]. \quad (49)$$

Denote the “AC factor” in the square brackets by A_s . Given that this expressions will only be used under the expectation $\langle \dots \rangle$, which effectively symmetrizes it over permutations of replicas, one can use any integer larger than s in defining this factor, and in particular one can replace A_s by A_{2n-2} .

Overall, by inserting into (47) we can write the coefficient E_n^{SK} as

$$E_n^{\text{SK}} = \frac{1}{n!} \sum_{s=2}^{2n-2} \frac{(-1)^s}{s} \sum_{\text{o.p.} \leq s}^{(s)} q_{a_1 a_2}^2 \cdots q_{a_{2n-3} a_{2n-2}}^2 A_{2n-2} \quad (50)$$

and so by comparison with (46) we get the elegant recursion

$$E_n^{\text{SK}} = \frac{1}{n} (E_{n-1}^{\text{SK}} + \frac{1}{2} \delta_{n-1,1}) A_{2n-2}. \quad (51)$$

Starting from $E_1 = \frac{1}{2}(q_{12}^2 - 1)$, this gives the explicit factorization (for $n \geq 2$)

$$E_n^{\text{SK}} = \frac{1}{2n!} q_{12}^2 A_2 A_4 \cdots A_{2n-2} \quad (52)$$

This shows clearly that the identities $\langle E_n^{\text{SK}} \rangle = 0$ for $n \geq 2$ are in fact all of AC type. Each such identity corresponds to the stochastic stability of the polynomial E_{n-1}^{SK} of the order below. Note that because we have already used permutation symmetry to rewrite the summation over a_1, \dots, a_{2n-2} in terms of E_{n-1} , the fully symmetric forms of the AC factors have to be maintained, e.g. in $A_2 = q_{12}^2 - 2(q_{13}^2 + q_{23}^2) + 3q_{34}^2$ one cannot replace q_{23}^2 by q_{13}^2 .

The explicit form of E_n^{SK} shows that only a subset of all AC identities is found from the energy term expansion: one has only one E_n for each n (whereas from $n = 3$ upwards there are more stochastically stable monomials that one can use in place of E_{n-1} to produce different AC identities), and functions of odd order like $q_{12}q_{13}q_{23}$ are missing altogether.

5 Expansion in the VB model

In the VB model the perturbation Hamiltonian is

$$\hat{H}(\tau) = \sum_{\nu=1}^{P_{\bar{\alpha}}} \hat{J}_{\nu} \tau_{i_{\nu}} \tau_{j_{\nu}}, \quad (53)$$

where $P_{\bar{\alpha}}$ is a Poisson variable of mean $\bar{\alpha}$, i_{ν} and j_{ν} for each ν are distributed uniformly over $\{1, \dots, M\}$, and \hat{J}_{ν} for each ν is ± 1 with equal probability. The expectation in the general expansion (38) is then

$$\begin{aligned} \langle \hat{H}(\tau^{a_1}) \cdots \hat{H}(\tau^{a_{2n}}) \rangle &= \\ &= \mathbb{E}_{P_{\bar{\alpha}}} \sum_{1 \leq \nu_1, \dots, \nu_{2n} \leq P_{\bar{\alpha}}} \mathbb{E}(\hat{J}_{\nu_1} \cdots \hat{J}_{\nu_{2n}}) \langle \tau_{i_{\nu_1}}^{a_1} \tau_{j_{\nu_1}}^{a_1} \cdots \tau_{i_{\nu_{2n}}}^{a_{2n}} \tau_{j_{\nu_{2n}}}^{a_{2n}} \rangle. \end{aligned} \quad (54)$$

The average over the \hat{J} vanishes except when the ν_1, \dots, ν_{2n} coincide in pairs or larger groups of even size, in which case it equals unity. The different patterns of groups that can occur are precisely the even integer partitions of $2n$, i.e. the integer partitions of n multiplied by two. We characterize such a partition of n by the number of times k_p each integer p occurs, such that $n = \sum_p p k_p$ (where the sum over p runs, here and in the following, from 1 to n). For $n = 3$, for example, the three different partitions are $(k_1, k_2, k_3) = (3, 0, 0)$, $(1, 1, 0)$ and $(0, 0, 1)$, corresponding to $3 = 1 + 1 + 1$, $3 = 1 + 2$, $3 = 3$. These correspond

respectively (after multiplication by two) to there being three pairs of distinct ν 's, one pair and one group of four, and one group of six ($\nu_1 = \dots = \nu_6$). For each partition, there are $(2n)!/(\prod_p (2p)!^{k_p} k_p!)$ possibilities — remember that the group sizes in the partition of $2n$ are $2p$, not p — of assigning ν_1, \dots, ν_{2n} and the corresponding replica indices a_1, \dots, a_{2n} to groups of the relevant sizes. Finally, given that each partition contains $g = \sum_p k_p$ different groups, there are $P_{\bar{\alpha}}(P_{\bar{\alpha}} - 1) \dots (P_{\bar{\alpha}} - g + 1) = P_{\bar{\alpha}}!/(P_{\bar{\alpha}} - g)!$ ways of assigning a value of ν to each group. Putting everything together gives, if $\sum_{(k)}$ denotes a sum over all distinct integer partitions (k_1, \dots, k_n) of n ,

$$\begin{aligned} \langle \hat{H}(\tau^{a_1}) \dots \hat{H}(\tau^{a_{2n}}) \rangle &= \\ &= \sum_{(k)} \frac{(2n)!}{\prod_p (2p)!^{k_p} k_p!} \mathbb{E}_{P_{\bar{\alpha}}} [P_{\bar{\alpha}}!/(P_{\bar{\alpha}} - g)!] \langle \tau_{i_1}^{a_1} \tau_{j_1}^{a_1} \dots \tau_{i_g}^{a_{2n}} \tau_{j_g}^{a_{2n}} \rangle \\ &= \sum_{(k)} \frac{(2n)! \bar{\alpha}^g}{\prod_p (2p)!^{k_p} k_p!} \langle q_{a_1 a_2 \dots}^2 \times \dots \times q_{\dots a_{2n-1} a_{2n}}^2 \rangle. \end{aligned} \quad (55)$$

Here the subscripts in the overlaps are arranged in accordance with the specific partition considered, e.g. for $n = 3$ and $(k) = (3, 0, 0)$ — corresponding to $6 = 2 + 2 + 2$ — the overlap product is $q_{a_1 a_2}^2 q_{a_3 a_4}^2 q_{a_5 a_6}^2$ while for $(k) = (1, 1, 0)$, which corresponds to $6 = 4 + 2$, it is $q_{a_1 a_2 a_3 a_4}^2 q_{a_5 a_6}^2$. In the line above, the subscripts (i_1, j_1) to (i_g, j_g) are arranged similarly, e.g. for $(k) = (1, 1, 0)$ the first four replicas (or more precisely replica indices) have subscripts (i_1, j_1) and the last two have subscripts (i_2, j_2) . Note that in this way we have picked out one particular assignment of replica indices to the groups of the partition, and multiplied accordingly with the number $(2n)!/(\prod_p (2p)!^{k_p} k_p!)$ of such assignments. This is on the understanding that the quenched average we are considering is to be used inside a symmetric sum over a_1, \dots, a_{2n} (to get the correct expression for a single setting of these summation variables we would need to symmetrize by averaging over all permutations $a_{\pi(1)}, \dots, a_{\pi(2n)}$).

To proceed, one inserts (55) into the general expansion (38). For the terms with $n \geq 2$ one could, as in the SK case, switch to sums over ordered pairs, but this is not as useful here as it does not prevent reductions in the order of the overlaps, e.g. we would still get $q_{1212}^2 = 1$. We therefore leave the sum unrestricted and write

$$e^{\text{VB}} = - \sum_{n \geq 1} \beta'^{2n} \sum_{g=1}^n \bar{\alpha}^g \langle E_{ng}^{\text{VB}} \rangle, \quad (56)$$

where the coefficient E_{ng}^{VB} is a sum over all integer partitions (k) of n with g terms:

$$E_{ng}^{\text{VB}} = \sum_{(k): g = \sum_{\alpha} k_{\alpha}} \frac{1}{\prod_p k_p!} E_{(k)} \quad (57)$$

$$\begin{aligned} E_{(k)} &= \frac{1}{(2n)! \prod_p (2p)!^{k_p}} \sum_{1 \leq a_1, \dots, a_{2n} \leq 2n} (-1)^m \times \\ &\times (m-1)!(2n-m)! q_{a_1 a_2 \dots}^2 \times \dots \times q_{\dots a_{2n-1} a_{2n}}^2 \end{aligned} \quad (58)$$

Let us discuss briefly how the above expression would change if we were considering Gaussian couplings \hat{J}_{ν} . Here, within a group of size p , one would

have $\mathbb{E}[\hat{J}_\nu^{2p}] = (2p)!/(p!2^p)$ instead of $= 1$ for the binary case. The only change for the case of Gaussian couplings is therefore that in (58) the factor $(2p)!^{k_p}$ is replaced by $(p!2^p)^{k_p}$.

We briefly compare (57,58) with the corresponding result (42) for the SK model, which after reinstating $\beta' \rightarrow \beta'(N/2)^{1/2}$ reads

$$e^{\text{SK}} = - \sum_{n \geq 1} (N\beta'^2/2)^n \langle E_n^{\text{SK}} \rangle. \quad (59)$$

In the VB case, to get the Boltzmann RaMOST we need to set $\bar{\alpha} = N\alpha$. Defining also $\tilde{\beta}^2 = 2\alpha\beta'^2$ as the equivalent SK temperature, the VB result (56) is

$$\begin{aligned} e^{\text{VB}} &= - \sum_{n \geq 1} (N\alpha\beta'^2)^n \sum_{g=1}^n (N\alpha)^{g-n} \langle E_{ng}^{\text{VB}} \rangle \\ &= - \sum_{n \geq 1} (N\tilde{\beta}^2/2)^n \sum_{g=1}^n (N\alpha)^{g-n} \langle E_{ng}^{\text{VB}} \rangle. \end{aligned} \quad (60)$$

We see that, as it should be, precisely the SK contribution (59) survives in the limit $\alpha \rightarrow \infty$ taken at fixed $\tilde{\beta}$: this is the contribution with the largest number $g = n$ of groups, where $(k) = (n, 0, \dots, 0)$ and the combinatorial factors are $1/\prod_p k_p! = 1/n!$ and $1/\prod_p (2p)!^{k_p} = 1/2^n$, giving $E_{nn}^{\text{VB}} = E_{(n,0,\dots,0)}/n! = E_n^{\text{SK}}$ by comparison with (43).

We can now write down the AC-like identities for the VB model that are obtained from the energy term (29) of the RaMOST. The r.h.s. of this exact expression contains only linear terms in $\bar{\alpha}$, so comparison with (56) shows that we must have $\langle E_{ng}^{\text{VB}} \rangle = 0$ for $g \geq 2$ (hence from $n \geq g$ also $n \geq 2$). In spite of the mixing of various orders of (multi-)overlaps, one can still get a recursive factorization of the $E_{(k)}^{\text{VB}}$ as we now show. Start from (58), and suppose the last overlap in the product is of order $2p$ (corresponding to a term of this size in an integer partition of $2n$). We want to do the sum over $a_{2(n-p)+1}, \dots, a_{2n}$. By analogy with the SK case, denote by s the size of the set $\{a_1, \dots, a_{2(n-p)}\}$. Introduce a sum over s , and a corresponding constrained sum over $a_1, \dots, a_{2(n-p)}$; in the latter, compress the summation range to $1, \dots, s$ and multiply by $(2n)!/[s!(2n-s)!]$ to make up for this. We get in this way

$$\begin{aligned} E_{(k)} &= \frac{1}{\prod_{p'} (2p')!^{k_{p'}}} \sum_{s=1}^{2(n-p)} \frac{1}{s!(2n-s)!} \times \\ &\times \sum_{\substack{(s) \\ 1 \leq a_1, \dots, a_{2(n-p)} \leq s}} q_{a_1 a_2 \dots}^2 \times \dots \times q_{\dots a_{2(n-p)-1} a_{2(n-p)}}^2 \\ &\times \sum_{1 \leq a_{2(n-p)+1}, \dots, a_{2n} \leq 2n} (-1)^m (m-1)! (2n-m)! q_{a_{2(n-p)+1} \dots a_{2n}}^2. \end{aligned} \quad (61)$$

We now focus on the sum in the last line; call it Σ . The complications in evaluating this arise because whenever a replica index occurs twice (or more) it cancels and we get a lower order overlap. So we need to consider again integer partitions, now of $2p$, to tell us how such identical indices group. Let $(\kappa) = (\kappa_1, \dots, \kappa_{2p})$ denote such a partition, with $\sum_r r\kappa_r = 2p$ (where $r = 1, \dots, 2p$

here and below). If replica indices occur in groups of identical values according to such a partition, all the even groups cancel completely from q , and we get a multi-overlap of order $\gamma_\emptyset = \sum_{r \text{ odd}} \kappa_r$ given by the number of odd groups. Define also $\gamma_e = \sum_{r \text{ even}} \kappa_r$, the number of even groups, and $\gamma = \gamma_\emptyset + \gamma_e$, the total number of groups. For every partition, there are $(2p)!/(\prod_r r!^{\kappa_r} \kappa_r!)$ ways of arranging the replica indices into groups of the given size.

Finally, we need to account for which actual replica index value in the range $1, \dots, 2n$ is used for each group of identical indices. All groups need to have distinct index values (since groups are *defined* as subsets of the summation variables $a_{2(n-p)+1}, \dots, a_{2n}$ having identical values). We split the sum over all possible assignments of index values according to the number of groups of odd size, $h_\emptyset \leq \gamma_\emptyset$, having *high* index values $> s$, and the number of even groups, $h_e \leq \gamma_e$, with such high index values. We need these quantities to determine the overall number m of distinct index values, given that the values $\{1, \dots, s\}$ occur already in the first line of (61): $m = s + h_\emptyset + h_e$. An explicit summation is required only over the $\gamma_\emptyset - h_\emptyset$ (low) index values $b_1, \dots, b_{\gamma_\emptyset - h_\emptyset}$ of the odd-sized groups that are in the range $1, \dots, s$. The other (high) index values of the odd groups we can set to $s+1, \dots, s+h_\emptyset$ by replica permutation symmetry. The index values of the even groups we never need explicitly because they disappear from the overlap. We take the $b_1, \dots, b_{\gamma_\emptyset - h_\emptyset}$ as ordered and then just need to work out how many index value assignments there are for a given setting of these indices: $\gamma_\emptyset!/[h_\emptyset!(\gamma_\emptyset - h_\emptyset)!]$ ways of choosing which odd groups have high index values, and similarly $\gamma_e!/[h_e!(\gamma_e - h_e)!]$ ways for the even groups; $(\gamma_\emptyset - h_\emptyset)!$ ways of permuting the $b_1, \dots, b_{\gamma_\emptyset - h_\emptyset}$ among the odd groups with low index values; $(2n - s)!/(2n - s - h_\emptyset)!$ ways of assigning index values to the odd groups with high indices (the order matters); and similar factors $[s - (\gamma_\emptyset - h_\emptyset)]!/[s - (\gamma_\emptyset - h_\emptyset) - (\gamma_e - h_e)]!$ and $(2n - s - h_\emptyset)!/(2n - s - h_\emptyset - h_e)!$ for the number of ways of assigning index values to the even groups with low and high indices, respectively. Putting everything together, we have

$$\begin{aligned}
\Sigma &= \sum_{(\kappa)} \frac{(2p)!}{\prod_r r!^{\kappa_r} \kappa_r!} \sum_{h_\emptyset, h_e} (-1)^{s+h_\emptyset+h_e} (s+h_\emptyset+h_e-1)! (2n-s-h_\emptyset-h_e)! \\
&\quad \times \frac{\gamma_\emptyset!}{h_\emptyset!(\gamma_\emptyset-h_\emptyset)!} \frac{\gamma_e!}{h_e!(\gamma_e-h_e)!} (\gamma_\emptyset-h_\emptyset)! \frac{(2n-s)!}{(2n-s-h_\emptyset)!} \\
&\quad \times \frac{[s-(\gamma_\emptyset-h_\emptyset)]!}{[s-(\gamma_\emptyset-h_\emptyset)-(\gamma_e-h_e)]!} \frac{(2n-s-h_\emptyset)!}{(2n-s-h_\emptyset-h_e)!} \\
&\quad \times \sum_{1 \leq b_1 < \dots < b_{\gamma_\emptyset-h_\emptyset} \leq s} q_{b_1 \dots b_{\gamma_\emptyset-h_\emptyset}, s+1, \dots, s+h_\emptyset}^2 \\
&= \sum_{(\kappa)} \frac{(2p)!(2n-s)!}{\prod_r r!^{\kappa_r} \kappa_r!} \sum_{h_\emptyset} (-1)^{s+h_\emptyset} \frac{\gamma_\emptyset! [s-(\gamma_\emptyset-h_\emptyset)]!}{h_\emptyset!} \\
&\quad \times \sum_{h_e} \frac{\gamma_e!}{h_e!(\gamma_e-h_e)!} (-1)^{h_e} \frac{(s+h_\emptyset+h_e-1)!}{[s-(\gamma_\emptyset-h_\emptyset)-(\gamma_e-h_e)]!} \\
&\quad \times \sum_{1 \leq b_1 < \dots < b_{\gamma_\emptyset-h_\emptyset} \leq s} q_{b_1 \dots b_{\gamma_\emptyset-h_\emptyset}, s+1, \dots, s+h_\emptyset}^2 \tag{62}
\end{aligned}$$

The sum over h_e can now be done, using that for $a \geq b$

$$\sum_{k=0}^n \frac{n!}{k!(n-k)!} (-1)^k \frac{(a+k)!}{(b+k)!} = (-1)^n \frac{(a-b)!a!}{(a-b-n)!(b+n)!} \quad (63)$$

(Factorials are treated like the corresponding Gamma functions here, i.e. when $a-b-n$ is negative, $(a-b-n)!$ is infinite and the result vanishes.) In our case $k \equiv h_e$ with $n = \gamma_e$, $a = s + h_\emptyset - 1$, $b = s + h_\emptyset - \gamma_\emptyset - \gamma_e$ so that

$$\begin{aligned} \Sigma &= \sum_{(\kappa)} \frac{(2p)!(2n-s)!}{\prod_r r!^{\kappa_r} \kappa_r!} \sum_{h_\emptyset} (-1)^{s+h_\emptyset} \frac{\gamma_\emptyset! [s - (\gamma_\emptyset - h_\emptyset)]!}{h_\emptyset!} \\ &\quad \times (-1)^{\gamma_e} \frac{(\gamma_\emptyset + \gamma_e - 1)!(s + h_\emptyset - 1)!}{(\gamma_\emptyset - 1)!(s + h_\emptyset - \gamma_\emptyset)!} \\ &\quad \times \sum_{1 \leq b_1 < \dots < b_{\gamma_\emptyset - h_\emptyset} \leq s} q_{b_1 \dots b_{\gamma_\emptyset - h_\emptyset}, s+1, \dots, s+h_\emptyset}^2 \\ &= \sum_{(\kappa)} \frac{(2p)!(-1)^s (s-1)!(2n-s)!}{\prod_r r!^{\kappa_r} \kappa_r!} (-1)^{\gamma_e} \gamma_\emptyset (\gamma_\emptyset + \gamma_e - 1)! A_s^{(\gamma_\emptyset)} \end{aligned} \quad (64)$$

with the higher-order AC factor (note $A_s^{(2)} \equiv A_s$)

$$A_s^{(\gamma_\emptyset)} = \sum_{h_\emptyset=0}^{\gamma_\emptyset} (-1)^{h_\emptyset} \frac{(s + h_\emptyset - 1)!}{h_\emptyset! (s-1)!} \sum_{1 \leq b_1 < \dots < b_{\gamma_\emptyset - h_\emptyset} \leq s} q_{b_1 \dots b_{\gamma_\emptyset - h_\emptyset}, s+1, \dots, s+h_\emptyset}^2 \quad (65)$$

Regarding the summation range for h_\emptyset in this definition, note that in (64) the factorials give the restriction that $s + h_\emptyset - \gamma_\emptyset \geq 0$, hence $h_\emptyset \geq \gamma_\emptyset - s$. This is ensured if in (65) we assign the sum the value zero when there is no possible ordered assignment of the summation variables because there are more than s such variables. For $\gamma_\emptyset = 0$, on the other hand, $h_\emptyset = 0$ also and so there are *no* summation variables in (65). In this case the sum can be set to unity, but in fact because of the factor γ_\emptyset in (65) we never need to evaluate $A_s^{(0)}$.

The factor $(s-1)!$ in (65), which diverges for $s = 0$, signals that this case needs to be checked separately. Because $s = |\{a_1, \dots, a_{2(n-p)}\}|$ this case can occur only if $n = p$, in which case $s = 0$ is the *only* possible value. The apparent divergence in the $(s-1)!$ factors can be traced back to the factor $(s + h_\emptyset + h_e - 1)!$ in (62), which looks divergent for $h_\emptyset = h_e = 0$. However, when $s = 0$, we must have $h_\emptyset = \gamma_\emptyset$, $h_e = \gamma_e$ and Σ from (62) simplifies to

$$\Sigma_0 = (2p)! \sum_{(\kappa)} \frac{(2p)!}{\prod_r r!^{\kappa_r} \kappa_r!} (-1)^{\gamma_\emptyset + \gamma_e} (\gamma_\emptyset + \gamma_e - 1)! q_{1 \dots \gamma_\emptyset}^2. \quad (66)$$

Because $\gamma_\emptyset + \gamma_e = g \geq 1$, the potentially offending factorial does indeed stay finite. Note that for this $s = 0$ expression the $\gamma_\emptyset = 0$ term does not vanish automatically as was the case for $s \geq 1$. The factor $(-1)^{\gamma_\emptyset}$ can be dropped because γ_\emptyset must be even (since $2p$ is). Note that (66) can be regarded as a special case of (65) if in the latter one first cancels the factor $(s-1)!$ (inserted into $A_s^{(\gamma_\emptyset)}$ for $s \geq 1$ to get a simple form), then sets $s = 0$ and $n = p$, and finally cancels $\gamma_\emptyset(h_\emptyset - 1)!/h_\emptyset! = 1$ because $h_\emptyset = \gamma_\emptyset$.

To summarize, the sum Σ takes the form (65) when $s \geq 1$, which is always the case for $p < n$, while for $s = 0$ and hence $n = p$ it is given by (66). We now just need to replace the last line of (61) by this to find, for $p < n$,

$$E_{(k)} = \frac{1}{\prod_{p'} (2p')!^{k_{p'}}} \sum_{s=1}^{2(n-p)} \frac{(-1)^s}{s} \sum_{1 \leq a_1, \dots, a_{2(n-p)} \leq s}^{(s)} q_{a_1 a_2 \dots}^2 \times \dots \times \quad (67)$$

$$\times q_{\dots a_{2(n-p)-1} a_{2(n-p)}}^2 \times \sum_{(\kappa)} \frac{(2p)!}{\prod_r r!^{\kappa_r} \kappa_r!} (-1)^{\gamma_e} \gamma_\emptyset (\gamma_\emptyset + \gamma_e - 1)! A_s^{(\gamma_\emptyset)}$$

while for $n = p$ and hence $(k) = (0, \dots, 0, 1)$

$$E_{(0, \dots, 0, 1)} = \sum_{(\kappa)} \frac{1}{\prod_r r!^{\kappa_r} \kappa_r!} (-1)^{\gamma_e} (\gamma_\emptyset + \gamma_e - 1)! q_{1 \dots \gamma_\emptyset}^2. \quad (68)$$

To make (67) into a recursion just takes a few more steps now. Call the sum over partitions in the second line, without the factor $(2p)!$,

$$B_s^{(2p)} = \sum_{(\kappa)} \frac{1}{\prod_r r!^{\kappa_r} \kappa_r!} (-1)^{\gamma_e} \gamma_\emptyset (\gamma_\emptyset + \gamma_e - 1)! A_s^{(\gamma_\emptyset)} \quad (69)$$

One now shows that, as for the second order case $\gamma_\emptyset = 2$ and because of permutation symmetry among replicas with indices $> s$, also in the higher order AC factors $A_s^{(\gamma_\emptyset)}$ one can replace s by any larger integer and in particular by $2(n-p)$. The same replacement can then be made in $B_s^{(2p)}$. In the first line of (67), one re-expands the summation range on the $a_1, \dots, a_{2(n-p)}$, but now only to $1, \dots, 2n-2p$ and correspondingly divides by $(2n-2p)!/[s!(2n-2p-s)!]$. The sum over s and the constraint of having s distinct summation indices can then be combined into an unconstrained sum, where $s = |\{a_1, \dots, a_{2(n-p)}\}|$:

$$E_{(k)} = \frac{1}{(2n-2p)!} \frac{1}{\prod_{p'} (2p')!^{k_{p'} - \delta_{p'p}}} \times \sum_{1 \leq a_1, \dots, a_{2(n-p)} \leq 2n-2p} (-1)^s (s-1)! (2n-2p-s)! \times q_{a_1 a_2 \dots}^2 \times \dots \times q_{\dots a_{2(n-p)-1} a_{2(n-p)}}^2 B_{2n-2p}^{(2p)}. \quad (70)$$

Comparison with (58) now shows that there is again a simple recursion:

$$E_{(k)} = E_{(\dots, k_p - 1, \dots)} B_{2n-2p}^{(2p)} \quad (71)$$

and starting from (68) every $E_{(k)}$ can be expressed in factorized form. The main difference between the VB and SK cases is that the factors entering at each step of the recursion are a mixture of AC factors of different orders (from 2 to $2p$). Also the final coefficient (57) at a given order is a sum over a number of factorized expressions, one for each even integer partition of $2n$ containing the specified number g of groups.

To get explicit expressions for the lowest order coefficients E_{ng}^{VB} we just need the initial values from (68) and the factors $B_s^{(2p)}$. We start with the former: for

$p = 1$, there are only two different partitions of $2p = 2$:

$$\begin{array}{ccccc}
(\kappa) & \gamma_\emptyset & \gamma_e & \prod_r r!^{\kappa_r} \kappa_r! & (-1)^{\gamma_e} (\gamma_\emptyset + \gamma_e - 1)! \\
(0, 1) & 0 & 1 & 2 & -1 \\
(2, 0) & 2 & 0 & 2 & 1
\end{array} \quad (72)$$

so

$$E_{(1)} = \frac{1}{2}(-1 + q_{12}^2) \quad (73)$$

as expected because this polynomial should equal E_1^{SK} . For $p = 2$ one gets similarly

$$\begin{array}{ccccc}
(\kappa) & \gamma_\emptyset & \gamma_e & \prod_r r!^{\kappa_r} \kappa_r! & (-1)^{\gamma_e} (\gamma_\emptyset + \gamma_e - 1)! \\
(0, 0, 0, 1) & 0 & 1 & 24 & -1 \\
(0, 2, 0, 0) & 0 & 2 & 8 & 1 \\
(1, 0, 1, 0) & 2 & 0 & 6 & 1 \\
(2, 1, 0, 0) & 2 & 1 & 4 & -2 \\
(4, 0, 0, 0) & 4 & 0 & 24 & 6
\end{array} \quad (74)$$

and thus

$$E_{(0,1)} = \left(-\frac{1}{24} + \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{2}{4}\right) q_{12}^2 + \frac{6}{24} q_{1234}^2 = \frac{1}{12}(1 - 4q_{12}^2 + 3q_{1234}^2). \quad (75)$$

For the first B -factor $B_s^{(2)}$ we again need the integer partitions of $2p = 2$:

$$\begin{array}{ccccc}
(\kappa) & \gamma_\emptyset & \gamma_e & \prod_r r!^{\kappa_r} \kappa_r! & (-1)^{\gamma_e} \gamma_\emptyset (\gamma_\emptyset + \gamma_e - 1)! \\
(0, 1) & 0 & 1 & 2 & 0 \\
(2, 0) & 2 & 0 & 2 & 2
\end{array} \quad (76)$$

so

$$B_s^{(2)} = A_s^{(2)}. \quad (77)$$

The first nontrivial B -factor is the one for $p = 2$, where we need integer partitions of 4:

$$\begin{array}{ccccc}
(\kappa) & \gamma_\emptyset & \gamma_e & \prod_r r!^{\kappa_r} \kappa_r! & (-1)^{\gamma_e} \gamma_\emptyset (\gamma_\emptyset + \gamma_e - 1)! \\
(0, 0, 0, 1) & 0 & 1 & 24 & 0 \\
(0, 2, 0, 0) & 0 & 2 & 8 & 0 \\
(1, 0, 1, 0) & 2 & 0 & 6 & 2 \\
(2, 1, 0, 0) & 2 & 1 & 4 & -4 \\
(4, 0, 0, 0) & 4 & 0 & 24 & 24
\end{array} \quad (78)$$

so that

$$B_s^{(4)} = \left(\frac{2}{6} - \frac{4}{4}\right) A_s^{(2)} + \frac{24}{24} A_s^{(4)} = -\frac{2}{3} A_s^{(2)} + A_s^{(4)}. \quad (79)$$

One sees that the terms with the highest-order multi-overlap follow a simple pattern: in $E_{(0,\dots,0,1)}$, $q_{1\dots 2p}^2$ has a prefactor of $1/(2p)$, while in the expression for $B_s^{(2p)}$, the AC polynomial $A_s^{(2p)}$ occurs with unit coefficient. Both of these observations follow from the fact that for $\gamma_\emptyset = 2p$ there are no even groups ($\gamma_e = 0$) and the partition of $2p$ has to be $(\kappa) = (2p, 0, \dots, 0)$. In (68) one has then $(-1)^{\gamma_e} (\gamma_\emptyset + \gamma_e - 1)! / \prod_r r!^{\kappa_r} \kappa_r! = (2p - 1)! / (2p)! = 1/(2p)$, while in (69), $(-1)^{\gamma_e} \gamma_\emptyset (\gamma_\emptyset + \gamma_e - 1)! / \prod_r r!^{\kappa_r} \kappa_r! = 2p(2p - 1)! / (2p)! = 1$.

Equipped with the above initial values, the recursion (71) and the definition (57) we can now get the relevant VB coefficients up to $n = 4$:

$$E_{11}^{\text{VB}} = E_{(1)} = \frac{1}{2}(-1 + q_{12}^2) = E_1^{\text{SK}} \quad (80)$$

$$E_{21}^{\text{VB}} = E_{(0,1)} = \frac{1}{12}(1 - 4q_{12}^2 + 3q_{1234}^2) \quad (81)$$

$$E_{22}^{\text{VB}} = \frac{1}{2}E_{(2,0)} = \frac{1}{2}E_{(1)}B_2^{(2)} = \frac{1}{4}(-1 + q_{12}^2)A_2^{(2)} = E_2^{\text{SK}} \quad (82)$$

$$E_{31}^{\text{VB}} = E_{(0,0,1)} = \dots + \frac{1}{6}q_{123456}^2 \quad (83)$$

$$E_{32}^{\text{VB}} = E_{(1,1,0)} = E_{(1)}B_2^{(4)} = \frac{1}{6}(-1 + q_{12}^2)(-2A_2^{(2)} + 3A_2^{(4)}) \quad (84)$$

$$\text{or} = E_{(0,1)}B_4^{(2)} = \frac{1}{12}(1 - 4q_{12}^2 + 3q_{1234}^2)A_4^{(2)} \quad (85)$$

$$E_{33}^{\text{VB}} = \frac{1}{6}E_{(3,0,0)} = \frac{1}{6}E_{(2,0)}B_4^{(2)} = \frac{1}{6}E_{(1)}B_2^{(2)}B_4^{(2)} \quad (86)$$

$$= \frac{1}{12}(-1 + q_{12}^2)A_2^{(2)}A_4^{(2)} = E_3^{\text{SK}} \quad (87)$$

$$E_{41}^{\text{VB}} = E_{(0,0,0,1)} = \dots + \frac{1}{8}q_{12345678}^2 \quad (88)$$

$$E_{42}^{\text{VB}} = E_{(1,0,1,0)} + \frac{1}{2}E_{(0,2,0,0)} = E_{(0,0,1)}B_6^{(2)} + \frac{1}{2}E_{(0,1)}B_4^{(4)} \quad (89)$$

$$= \left(\dots + \frac{1}{6}q_{123456}^2 \right) A_6^{(2)} + \frac{(1 - 4q_{12}^2 + 3q_{1234}^2)}{24} \left(-\frac{2}{3}A_4^{(2)} + A_4^{(4)} \right) \quad (90)$$

$$E_{43}^{\text{VB}} = \frac{1}{2}E_{(2,1,0,0)} = \frac{1}{2}E_{(1,1,0)}A_6^{(2)} = \frac{1}{2}E_{(0,1)}A_4^{(2)}A_6^{(2)} \quad (91)$$

$$= \frac{1}{24}(1 - 4q_{12}^2 + 3q_{1234}^2)A_4^{(2)}A_6^{(2)} \quad (92)$$

$$E_{44}^{\text{VB}} = \frac{1}{24}E_{(4,0,0,0)} = \frac{1}{48}(-1 + q_{12}^2)A_2^{(2)}A_4^{(2)}A_6^{(2)} = E_4^{\text{SK}} \quad (93)$$

Note that in all terms with $g \geq 2$, which are the ones we are interested in because they give us the identities $\langle E_{ng}^{\text{VB}} \rangle = 0$, the constant contribution in the factor $E_{(0,\dots,0,1)}$ can be dropped. E.g. in E_{33}^{VB} , we can symmetrize in $(-1/12)A_2^{(2)}A_4^{(2)}$ across the replica indices 1, 2, 3, 4 because $A_4^{(2)}$ is symmetric in these indices anyway. But under this symmetrization $A_2^{(2)}$ vanishes. We have already exploited this in writing $E_{22}^{\text{VB}} = E_2^{\text{SK}}$, $E_{33}^{\text{VB}} = E_3^{\text{SK}}$ and $E_{44}^{\text{VB}} = E_4^{\text{SK}}$ above.

Now consider the various identities that result in detail: $\langle E_{22}^{\text{VB}} \rangle = 0$ is, after dropping the constant in the first factor, the standard 4th order AC relation $\langle q_{12}^2 A_2^{(2)} \rangle = 0$ [8]. Next, $\langle E_{33}^{\text{VB}} \rangle = 0$ is, after dropping the -1 , the same identity as for the SK model, $\langle q_{12}^2 A_2^{(2)} A_r^{(2)} \rangle = 0$. The relation $\langle E_{32}^{\text{VB}} \rangle = 0$ reduces to $\langle q_{1234}^2 A_4^{(2)} \rangle = 0$ (the constant can be dropped, and $\langle q_{12}^2 A_4^{(2)} \rangle = \langle q_{12}^2 A_2^{(2)} \rangle = 0$) as in [1][22]. In the first form written down above one can similarly reduce everything to $\langle q_{12}^2 A_2^{(4)} \rangle = 0$, which must — and does indeed — give an equivalent relation, but looks superficially different because we have broken the replica permutation symmetry in a different manner. However, this simple pattern (of $\langle q_{1\dots 2n} A_{2n}^{(2)} \rangle = 0$) does not persist to higher orders, as the $n = 4$, $g = 2$ term shows: this is the first one where one gets a sum over several partitions.

After dropping terms that are zero because of lower order identities, $\langle E_{42}^{\text{VB}} \rangle = 0$ becomes

$$\frac{1}{24} \langle 4q_{123456}^2 A_6^{(2)} + 3q_{1234}^2 A_4^{(4)} \rangle = 0, \quad (94)$$

and the two parts cannot be separated, at least not provably so from the energy term expansion considered here.

In $\langle E_{43}^{\text{VB}} \rangle$, finally, the q_{12}^2 -term can be dropped because

$$q_{12}^2 A_4^{(2)} A_6^{(2)} = q_{12}^2 A_2^{(2)} A_4^{(2)}$$

which has vanishing expectation due to the identity from E_{33}^{VB} . So one gets

$$\langle q_{1234}^2 A_4^{(2)} A_6^{(2)} \rangle = 0 \quad (95)$$

$\langle E_{44}^{\text{VB}} \rangle = 0$, finally, gives the 8-th order SK identity $\langle q_{12}^2 A_2^{(2)} A_4^{(2)} A_6^{(2)} \rangle = 0$.

6 Outlook

The work presented in this paper was motivated by recent progress [5, 32, 33] in our understanding of relations among ultrametricity [29] and polynomial identities [1, 22] in mean field spin glasses. We first reviewed the concept of random overlap structures [3], both for fully connected and for diluted disordered mean field spin systems. Then, starting from an explicit expression for the *energy* within this framework, we compared this to an expansion closer to the ones obtained by stochastic stability [15][34] or smooth cavity field [6] methods. We analysed the resulting linear set of overlap identities (which usually develop in statistical mechanics of quenched disordered systems), referred to as Aizenman-Contucci equations [1].

We extended previous results [7][8] both by deriving an alternative and more rigorous recursive approach for the derivation of these identities and by showing that, at least when considering the energy term of the Boltzmann ROST/RaMOST, the identities obtained from the low orders of our expansion are in perfect agreement with the same relations obtained with e.g. the replica trick. Going to higher orders in the expansion, on the other hand, we found that the resulting identities are fewer in number than the identities known to hold for the Sherrington-Kirkpatrick [22] or the Viana-Bray [21] models.

As the Parisi solution of the SK model (encoded in Ruelle's GREM [37] within this framework and called Parisi ROST) is known to satisfy the whole set of AC identities [35], our work strongly suggests that these further, missing relations must be associated with the entropic contribution of the ROST, on which we plan to report soon.

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